Linear Syitem Theory – Lecture 5

Mc Millan form – Structure Theorem

Ph. Müllhaupt
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Outline:

1. McMillan form (Smith-McMillan)
2. Test for unimodularity
3. Controllable companion form   state-space
   observable   "   "
4. Structure theorem
5. Transfer Matrix and Realizations

1. McMillan form

given a transfer matrix $G(s)$ $\updownarrow q$ $\underset{p}{\longleftrightarrow}$

1°/ Let $d(s)$

l.c.d. (least common denominator)

of elements of $G(s)$

$$G(s) \stackrel{\Delta}{=} N(s) \Big/ d(s)$$

$N(s)$ is a polynomial matrix

2°/ Perform a Smith decomposition of $N(s)$

$$\underset{\substack{\nearrow\\ \text{unimodular}}}{L}\; N \;\underset{\substack{\uparrow\\ \text{unimodular}}}{R} \;\triangleq\; \underset{\substack{\nwarrow\\ \text{diagonal.}}}{S}$$

3°/ compute $\dfrac{S}{d}$ and cancel common factors

$$M(s) \triangleq \frac{S}{d} = \begin{cases} \left(\operatorname{diag}\left(\varepsilon_i / \psi_i\right),\ 0_{n,p-q}\right) & p>q \\ \operatorname{diag}\left(\varepsilon_i / \psi_i\right) & p=q \\ \begin{pmatrix} \operatorname{diag}\left(\varepsilon_i / \psi_i\right) \\ 0_{q-p,p} \end{pmatrix} & p<q \end{cases}$$

Example

$$G(s) = \begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{1}{(s+1)(s+2)} \\[2ex] \dfrac{1}{(s+1)(s+2)} & \dfrac{s+3}{(s+2)^2} \end{bmatrix}$$

$$d(s) = (s+1)^2 (s+2)^2$$

$$N(s) = \begin{bmatrix} (s+2)^2 & (s+1)(s+2) \\ (s+1)(s+2) & (s+1)^2 (s+3) \end{bmatrix}$$

$$S(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)^2(s+2)^3 \end{bmatrix}$$

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & s+2 \end{bmatrix}$$

Remark: Since each invariant polynomial in the Smith form of $S(s)$ divides non-zero polynomials that succeeds it, it follows that if a common factor is cancelled when forming the ε_i, the same factor is cancelled when forming each succeeding ε_j

From this $\psi_1, \psi_2, \dots, \psi_{i-1}$ $i = 2, 3 \dots, q$

where ε_q is the last non-zero numerator.

The divisors property of the invariant polynomials in S is retained

$\mathcal{E}_i$ divides $\mathcal{E}_{i+1}, \mathcal{E}_{i+2}, \dots, \mathcal{E}_q$

$i = 1, \dots, q-1.$

$S+2$ is regarded as $(S+2)/1$

$\mathcal{E}_1 = 1 \qquad \mathcal{E}_2 = S+2$

$\psi_2 = 1 \qquad \psi_1 = (S+1)^2(s+2)^2$

Theorem:

$$P(s) = \begin{bmatrix} T & U \\ -V & W \end{bmatrix} \quad (r+p) \times (r+q)$$

Smith form of $P \longrightarrow S_P$

Smith form of $T \longrightarrow S_T$

$$S_p = \begin{bmatrix} I_r & O_{r,p} \\ O_{q,r} & \mathrm{diag}(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_p) \end{bmatrix}$$

$$S_T = \begin{bmatrix} I_{r-p} & O_{r-p} \\ O_{p,r-p} & \mathrm{diag}\,(\psi_p, \psi_{p-1}, \dots, \psi_1) \end{bmatrix}$$

(above is the case when $p=q$, as many inputs as outputs)

2. Test for unimodularity

Perform a Smith decomposition

unimodular $\iff$ $S = I$

3. Controllability / observability : companion forms.

State-space
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Controllability: given any time T,
the system is controllable when it is possible
to steer the system from arbitrary $x_0 \in \mathbb{R}^n$

$x_0 = x(0)$ to an arbitrary $x(T) \in \mathbb{R}^n$.

Observability

($\equiv$ detectability for linear time.inv. systems)

The system is observable if setting $u=0$, it is possible to deduce x_0 from $\{y(t)\}$.

rank conditions for contrl./obs.

$$\mathcal{C} = {\scriptstyle n}\updownarrow \underbrace{\begin{bmatrix} \overset{p}{\overleftrightarrow{B}} & AB & \cdots & A^{n-1}B \end{bmatrix}}_{n \times p} \qquad \text{(ctrl matrix)}$$

$$\mathcal{O} = \begin{bmatrix} \overset{\overleftrightarrow{\quad n \quad}}{C} \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{array}{l} \updownarrow q \\ \\ n \times q \\ \\ \end{array} \qquad \text{(obs. matrix)}$$

$$\begin{cases} \text{System is ctrl.} \iff \text{rank } \mathcal{C} = n \\ \quad \text{"} \quad \text{is obs.} \iff \quad \text{"} \quad \mathcal{O} = n \end{cases}$$

Controllable companion form

$$\dot{x} = Ax + Bu$$

Let us suppose $\operatorname{rank} \mathcal{C} = n$

does there exist a P such that

$\bar{A} = PAP^{-1}$ has a "canonical" form, where controllability is "evident".

observable companion form

can be deduced from the controllable companion form using duality

duality principle:

$$\underset{\text{controllable}}{(A, B)} \iff \underset{\text{observable}}{\begin{pmatrix} A^T \\ C = B^T \end{pmatrix}}$$

Computing the companion controllable form

Let us suppose B of full column rank.

$\dot{x} = Ax + Bu$ $\qquad B = [b_1, b_2, \dots, b_p]$

construct the $\mathcal{C}$ matrix and from it extract the following matrix

$$L = \Big[\underline{\underline{b_1}}, \underline{\underline{Ab_1}}, \underline{\underline{A^2 b_1}}, \dots, A^{d_1-1} b_1, b_2, Ab_2, \dots, A^{d_2-1} b_2, \dots, A^{d_p-1} b_p\Big]$$

$$\mathcal{C} = \Big[\underbrace{\underbrace{B}_{\longleftrightarrow\; b_1 \cdots b_p} \quad AB \quad \cdot \; A^{\bar{n}} B\big]}\; \cdots \quad A^{n-1}B\Big]$$

$\mathrm{rank}\,[B \cdots A^{\bar{n}} B] = n$

<u>L is $n \times n$.</u>

<u>Definition</u> $\qquad \sigma_k = \sum_{i=1}^{k} d_i \qquad k = 1, 2, \dots, p$

d_i are called the controllability indices.

$$\sigma_1 = d_1$$
$$\sigma_2 = d_1 + d_2$$
$$\sigma_3 = d_1 + d_2 + d_3$$

$$\sigma_p = d_1 + d_2 + \dots + d_p = n$$

Compute L^{-1}

and set q_k equal to the σ_k^{th} row of L^{-1} for $k = 1, 2, \dots, p$.

The similarity transform that will bring the system into controllable companion form is

$$P = \begin{bmatrix} q_1 \\ q_1 A \\ \vdots \\ q_1 A^{d_1 - 1} \\ q_2 \\ q_2 A \\ \vdots \\ q_2 A^{d_2 - 1} \\ \vdots \\ q_p A^{d_p - 1} \end{bmatrix}$$

$$\bar{A} = PAP^{-1} \qquad \bar{B} = PA$$

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \dots & \bar{A}_{1p} \\ \bar{A}_{p1} & & \dots & \bar{A}_{pp} \end{bmatrix}$$

$$\left[\begin{array}{c|c|c}
\begin{matrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ \times & \times & \times & \times & \times & \times \end{matrix} & \begin{matrix} O \\ \\ \\ \times & \times & \times & \times & \times \end{matrix} & \begin{matrix} O \\ \\ \\ \times & \times & \times & \times & \times \end{matrix} \\
\hline
O & \begin{matrix} 0 & 1 & 0 & & \\ & & \ddots & & O \\ O & & & & 1 \end{matrix} & O \\
\times\ \times\ \times\ \times\ \times & \times\ \times\ \times\ \times\ \times\ \times & \times\ \times\ \times\ \times\ \times\ \times \\
\hline
O & O & \begin{matrix} 0 & 1 & & O \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \ \ 1 \end{matrix} \\
\times\ \times\ \times\ \times & \times\ \times\ \times\ \times\ \times\ \times & \times\ \times\ \times\ \times\ \times\ \times
\end{array}\right]$$

$$\overline{B} = PB = \begin{array}{c} \\ d_1 \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \\ \\ d_2 \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \\ \\ \\ d_p \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 & & & \\ 0 & & O & \\ \vdots & & & \\ 1 & \times & \times & \times \\ 0 & 0 & & \\ 0 & 0 & O & \\ 0 & 0 & & \\ 0 & 1 & \times & \times \\ & \vdots & & 0 \\ & & & 0 \\ & O & & 0 \\ & & & 0 \\ & & & 0 \\ & & & 1 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 3 & 0 & -3 & 1 \\ -1 & 1 & 4 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$\bar{\mathcal{C}} = \begin{bmatrix} b_1 & b_2 & Ab_2 & A^2b_2 \\ & & & \end{bmatrix} = L$$

$$L = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & -3 & -10 \\ 0 & 1 & 4 & 13 \\ 0 & 0 & -1 & -3 \end{bmatrix}$$

$d_1 = 1$

$d_2 = 3$

$\sigma_1 = 1 = d_1$

$\sigma_2 = d_1 + d_2 = 4 = n$

L^{-1} and pick the 1st row $(\sigma_1 = 1)$

$$q_1 = \begin{bmatrix} 1 & 1 & 0 & -2 \end{bmatrix}$$

and pick the 4th row of L^{-1} $(\sigma_2 = 4)$

$$q_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} q_1 \\ q_2 \\ q_2 A \\ q_2 A^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\bar{A} = PAP^{-1} = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{array}\right]$$

$$\bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = PB$$

$\{\bar{A}, \bar{B}\}$ is the controllable companion form of $\{A, B\}$.

4. Structure theorem

$$\underbrace{\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}}_{\text{controllable companion form.}} = \{PAP^{-1}, PB, CP^{-1}, D\}$$

(p) controllability indices d_i for $i=1, \cdots, p$.

which specify the dimensions of the various diagonal companion-form submatrices of $\bar{A}$ as well as the (p) ordered integers

$$\sigma_k = \sum_{i=1}^{k} d_i \quad k = 1, 2, \dots, p$$

which denote the "non trivial" rows of $\bar{A}$ and $\bar{B}$.

We now define

$\bar{A}_p$ as the $(p \times n)$ matrix consisting of the (p) ordered σ_k rows of $\bar{A}$ (and $\bar{B}_p$ as the $(p \times p)$ matrix consisting of the (p) ordered σ_k rows of $\bar{B}$)

(Remark $\bar{B}_p$ is upper right triangular and hence invertible) $|B_p| = 1$

$\bar{A}_p$ assumes no particular special form.

define the S matrix (⚠ different from S of the Smith McMillan form)

$$S = \overset{\xleftrightarrow{\quad p \quad}}{\left[\begin{array}{ccccc}
1 & 0 & 0 & & \\
s & 0 & 0 & & \\
\vdots & \vdots & & & \\
s^{d_1-1} & 0 & 0 & & \\
\hline
0 & 1 & 0 & & \\
0 & s & 0 & & \\
\vdots & \vdots & & & \\
0 & s^{d_2-1} & 0 & \cdots & 0 \\
\hline
0 & 0 & 1 & & \\
\vdots & \vdots & s & & \\
0 & 0 & s^{d_3-1} & & \\
\hline
\vdots & \vdots & \vdots & & \vdots \\
\hline
 & & & & 1 \\
 & 0 & & \cdots & s \\
 & & & & \vdots \\
 & & & & s^{d_p-1}
\end{array}\right]}$$

The structure theorem

If a state representation $\{A, B, C, D\}$ is controllable with B full rank $p \leq n$, its transfer matrix $G(s)$ given $C\,(sI-A)^{-1}B + D$ can expressed as

$$G(s) = \bar{C}\; S(s)\, \delta^{-1}(s)\, \bar{B}_p + D$$

$$= \left[\bar{C}\, S(s) + D\, \bar{B}_p^{-1}\, \delta(s)\right]\left[\bar{B}_p^{-1}\, \delta(s)\right]^{-1}$$

$$\delta(s) = \begin{bmatrix} s^{d_1} & & 0 \\ & s^{d_2} & \\ 0 & & \ddots \\ & & & s^{d_p} \end{bmatrix} - \bar{A}_p \, S(s)$$

<u>*Example*</u>

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$PAP^{-1} = \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 9 & 1 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = PB$$

$$CP^{-1} = \bar{C} = \begin{bmatrix} 5 & 4 & 1 \\ -9 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$p = 1 \qquad d_1 = 3 = \sigma_1 = n$$

$$S(s) = \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} \qquad \begin{aligned} \bar{A}_p &= \begin{bmatrix} -9 & 9 & 1 \end{bmatrix} \\ \bar{B}_p &= 1 \end{aligned}$$

$$\begin{aligned} \delta(s) &= s^3 - \bar{A}_m \cdot S(s) \\ &= s^3 - s^2 - 9s + 9 = (s-1)(s+3)(s-3) \end{aligned}$$

$$G(s) = \bar{C} \; S \; \delta^{-1} \; \bar{B}_p + D =$$

$$\begin{bmatrix} -s^2+4s+5 \\ \\ s^2-9 \end{bmatrix} \cdot \frac{1}{(s-1)(s+3)(s-3)}$$

$$G(s) = \begin{bmatrix} \dfrac{(s+1)(5-s)}{(s-1)(s+3(s-3)} \\ \\ \dfrac{s}{s-1} \end{bmatrix}$$

The structure theorem enables us to express the transfer matrix $G(s)$ as the product of a $(q \times p)$ polynomial matrix and the inverse of another $(p \times p)$ polynomial matrix.

$$\bar{N} \triangleq \bar{C} \; S(s) + D \, \bar{B}_p^{-1} \; \delta(s)$$

$$\bar{M} \triangleq \bar{B}_p^{-1} \; \delta(s)$$

$$G(s) = \bar{N} \, \bar{M}^{-1}$$

$$\bar{N} = \bar{C}\, S(s) + D\bar{B}_p^{-1}\delta(s) = \begin{bmatrix} -s^2+4s+5 \\ s^3-9s \end{bmatrix}$$

$$= \begin{bmatrix} (s+1)(5-s) \\ s(s+3)(s-3) \end{bmatrix}$$

$$\bar{M} = \bar{B}_p^{-1}\delta(s) = s^3 - s^2 - 9s + 9$$

$$= (s-1)(s+3)(s-3)$$

$$\bar{N}\bar{M}^{-1} = G(s)$$

5. How to construct a realization (state-space form in controllable companion form) starting from the transfer matrix.

Example:

$$P = \left[\begin{array}{ccc|cc} I_2 & 0 & 0 & 0 & 0 \\ 0 & (s+1)^2(s+2) & 0 & s+2 & s+1 \\ 0 & -(s+1) & s+2 & 0 & 1 \\ \hline 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} T & U \\ \hline -V & W \end{array}\right]$$

$$G(s) = -VT^{-1}U + W = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+2)^2} \end{bmatrix}$$

McMillan Form

$$\varepsilon_1 = 1$$
$$\varepsilon_2 = s+2$$
$$\psi_1 = (s+1)^2(s+2)^2$$
$$\psi_2 = 1$$

Smith Form of T

$$\begin{bmatrix} I_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (s+1)^2(s+2)^2 \end{bmatrix}$$

Smith Form of P

$$\begin{bmatrix} I_4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s+2 \end{bmatrix}$$

Let us use the structure theorem to compute a companion controllable realization

1°/ l. c. d. (least common denominator) of each column of G(s)

$$g_1 = (s+1)^2(s+2) \qquad g_2 = (s+2)^2(s+1)$$

$$G(s) = \begin{bmatrix} r_{11}/g_1 & \cdots & r_{1p}/g_p \\ \vdots & & \vdots \\ r_{q1}/g_1 & \cdots & r_{qp}/g_p \end{bmatrix}$$

$$\operatorname{diag}(g_j) = \begin{bmatrix} (s+1)^2(s+2) & 0 \\ 0 & (s+2)^2(s+1) \end{bmatrix}$$

$$r = \begin{bmatrix} s+2 & s+2 \\ s+1 & (s+3)(s+1) \end{bmatrix}$$

so that $G(s) = r \cdot (\operatorname{diag}(g_j))^{-1}$

$$r = \begin{bmatrix} s^3+4s^2+5s+2 & 0 \\ 0 & s^3+5s^2+8s+4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} s^3 & 0 \\ 0 & s^3 \end{bmatrix}}_{\operatorname{diag}(s^{d_i})} + \underset{\overline{A_p}}{\begin{bmatrix} 2 & 5 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 5 \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 0 \\ s & 0 \\ s^2 & 0 \\ 0 & 1 \\ 0 & s \\ 0 & s^2 \end{bmatrix}}_{S(s)}$$

$d_1 = 3$

$d_2 = 3$

$$D = \lim_{s \to \infty} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = (r_{ij0})$$

Determine

in our case

$$\bar{C}\, S(s) = r_{-}\ (r_{iju})\, \mathrm{diag}(g_j\ )$$

$$\bar{C}\ S(s) = \begin{bmatrix} s+2 & s+2 \\ s+1 & s^2+4s+3 \end{bmatrix} = r$$

$$\bar{C} = \begin{bmatrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 3 & 4 & 1 \end{bmatrix}$$

$$\bar{A} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ -2 & -5 & -4 & 0 & 0 & 0 \\ \hline & & & 0 & 1 & 0 \\ & 0 & & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & -8 & -5 \end{array}\right] \qquad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 3 & 4 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{C}\,(sI - \bar{A})^{-1}\,\bar{B} + D = \underline{\underline{G(s)}}\,.$$